PM94-19 hep-th ymnnn

Variational solution of the Gross-Neveu model

I. The large-N limit

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(March 26, 1995)

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Abstract

In this first paper we begin the application of variational methods to renormalizable asymptotically free field theories, using the Gross-Neveu model as a laboratory. This variational method has been shown to lead to a numerically convergent sequence of approximations for the anharmonic oscillator. Here we perform a sample calculation in lowest orders, which show the *superficially* disastrous situation of variational calculations in quantum field theory, and how in the large-N limit all difficulties go away, as a warm up exercise for the finite-N case.

I. INTRODUCTION.

Many non-perturbative methods have been developed in recent years, trying to improve the asymptotic Taylor expansions (in terms of some coupling constant parameter) one usually encounters in quantum field theories. A series of elegant papers in the seventies has explored and clarified various aspects of the large order behavior of perturbation theory in quantum mechanical and quantum field theoretical systems. In some cases this newly acquired knowledge was successfully used to obtain more accurate results from the asymptotic perturbative series via Borel transforms. In other cases, especially in four-dimensional field theoretic models, the question of what the sum of a badly-behaved perturbation series means, remains unanswered [1].

Recently, there has been a considerable interest in variational-type expansions ([2]- [10]). Because of its conceptual simplicity and usually quite astonishing numerical accuracy, this type of expansion would be of particular importance in quantum field theoretical calculations. Despite the distinctly intuitive nature of such a procedure, there has recently appeared strong evidence that optimized perturbation theory may indeed lead to a rigorously convergent series of approximants even in strong coupling cases. In particular, the convergence of this variational-like procedure has been rigorously established in the case of "zero" and "one" dimensional field theories [11].

In this paper, we begin the application to quantum field theory of the variational improvement of perturbation theory as described in [12]. In the case of the anharmonic oscillator, this method gives a sequence of approximations which converges to correct answers as described in details in [13,14]. Here, as a laboratory, using dimensional regularization, we study the $O(2N)$ Gross-Neveu model [15] in $2+\varepsilon$ dimensions. Exact results are known in 2 dimensions for the mass gap [16] and the vacuum energy density [17] in terms of $\Lambda_{\overline{MS}}$ for all N.

Before comparing variational calculations to these exact results, one must first understand how to handle infinities at $\varepsilon \to 0$ in a way compatible with perturbative renormalization, leaving only finite physical quantities expressed only in terms of finite physical renormalization group invariants of the effective variational approximation. This is the only sensible procedure if we want the physical characteristics of the theory, which have led to the specific choice of the effective variational approximation, to survive renormalization.

In section II, at $\varepsilon \neq 0$, we compute the vacuum energy density and mass gap, according to the variationally improved perturbation theory of [12–14] in the few lowest orders, adding and subtracting a bare fermion mass, in closest analogy with the procedure which converges so remarkably in the anharmonic oscillator case. Not surprisingly, the procedure is in general highly singular as $\varepsilon \to 0$, when the perturbative order is fixed. However, for $N \to \infty$, the variational procedure gives the exact answers immediately when applied at first non trivial order.

In section III, using this remark and the fact that for $N \to \infty$ the theory reduces to cactus diagrams which can be summed in closed (if implicit) form, we show how to remove the singularity of the procedure as $\varepsilon \to 0$, obtaining finite, renormalization group invariant expressions for the vacuum energy and mass gap of the massless theory in term of the physical, finite, renormalization group invariant mass parameter of the effective massive theory, with respect to which we can then optimize. The optimum, which gives the exact known answers, turns out to be at zero, which is a physical consequence of the fact that one is actually solving the theory exactly. The procedure is sufficiently transparent to allow for a generalization to arbitrary N, which is done in the second paper.

II. LOW ORDER CALCULATIONS.

In $2+\varepsilon$ dimensions, it is a straightforward exercise to compute the vacuum graphs of Fig. 1, which are the only ones which survives in the large N limit, using the Lagrangian

$$\mathcal{L} = i\sum_{i=1}^{N}\bar{\psi}_i\partial\!\!\!/\psi_i + m_0\sum_{i=1}^{N}\bar{\psi}_i\psi_i + \frac{g_0^2}{2}(\sum_{i=1}^{N}\bar{\psi}_i\psi_i)^2 \quad , \tag{II.1}$$

where m_0 and g_0 are bare parameters (in the following, we shall suppress the summation over the index i). We use the bare parameters for several reasons. First, from the remarkable results for the anharmonic oscillator [13,14], we may conjecture that our procedure converges to the exact answer <u>for fixed $\varepsilon < 0$</u> for any N. Second, the use of renormalized parameters introduces counterterms, which in effect mix perturbative orders and confuses matters as we are already perturbing in m_0. It is thus preferable to perform as much of the calculation as possible with bare parameters, and only at the end relate them to the renormalized ones via the renormalization group through the standard procedure. The crucial requirement is of course that the final physical result remain finite for $\varepsilon \to 0$.

We give the value of the contribution of the graphs of Fig. 1 to the energy density E_0 for all N :

$$\begin{aligned} E_0\,(m_0) = {} & \frac{m_0^{2+\varepsilon} N\,\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\,[\,\frac{2}{2+\varepsilon} \;+\; (2N-1)\,g_0^2\,m_0^{\varepsilon}\,\frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}} \\ & + (2N-1)^2\,g_0^4\,m_0^{2\varepsilon}\,(1+\varepsilon)\,\frac{\Gamma^2(-\frac{\varepsilon}{2})}{(4\pi)^{2+\varepsilon}} \;+\; O(g_0^6)\,] \\ \equiv {} & E^{(0)}\,(m_0) \;+\; g_0^2\,E^{(1)}\,(m_0) \;+\; g_0^4\,E^{(2)}\,(m_0) \quad . \end{aligned} \tag{II.2}$$

The straightforward application of the variational procedure at lowest order calls for the minimization with respect to m_0 of the function

$$E_0^{(1)}\,(m_0) = E^{(0)}\,(m_0) \;+\; g_0^2\,E^{(1)}\,(m_0) \;-\; m_0\,\frac{\partial E^{(0)}\,(m_0)}{\partial m_0} \quad . \tag{II.3}$$

This gives the optimal value for the variational parameter

$$1 = (2N-1)\,\frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\,g_0^2\,m_{0(opt)}^{\varepsilon} \quad , \tag{II.4}$$

and the corresponding optimum :

$$E_0^{(1)} = m_{0(opt)}^{2+\varepsilon}\;N\,\frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\,\frac{\varepsilon}{2+\varepsilon} \quad . \tag{II.5}$$

When N goes to infinity, one can compute E_0 exactly using the effective potential for the field $\sigma = \bar{\psi}\psi$. In this large-N limit, one finds that $E_0^{(1)}$ coincides with the exact E_0 and m_0 with the exact mass gap for all values of ε . This fact has already been noted in Ref. [13] and corresponds to the fact that for $N = \infty$ the theory is a set of N free massive fermions, and the Hartree-Fock approximation is exact. We notice that for $N = \infty$, $g_0^2\,N$ fixed, m_0 and $E_0^{(1)}$ as given by Eqs. (II.4) and (II.5) are finite for $\varepsilon \to 0$ provided that the renormalized coupling g^2, given by

$$g_0^2 = \frac{g^2\,\mu^{-\varepsilon}}{1 - \frac{(N-1)\,g^2}{\pi\,\varepsilon}} \quad , \tag{II.6}$$

is kept fixed. However, $E^{(0)}$ and $g_0^2\,E^{(1)}$ are separately infinite.

The next order of the approximation calls for the minimization with respect to m_0 of

$$\begin{aligned} E_0^{(2)}\,(m_0) = {} & E^{(0)}\,(m_0) \;+\; g_0^2\,E^{(1)}\,(m_0) \;+\; g_0^4\,E^{(2)}\,(m_0) \\ & - m_0\,\frac{\partial E^{(0)}\,(m_0)}{\partial m_0} \;-\; g_0^2\,m_0\,\frac{\partial E^{(1)}\,(m_0)}{\partial m_0} \\ & + \frac{1}{2}\,m_0^2\,\frac{\partial^2 E^{(0)}\,(m_0)}{\partial m_0^2} \quad . \end{aligned} \tag{II.7}$$

Just as for the anharmonic oscillator, and for the same reasons, $E_0^{(2)}\,(m_0)$ has an extremum at the same value of m_0 [Eq. (II.4)] as $E_0^{(1)}\,(m_0)$, where it takes the same value [Eq. (II.5)], reflecting the fact that in the large-N limit, the variationally improved perturbative treatment gives the exact answer at each order.

Although the behaviour of the large-N limit is satisfactory, this sample calculation reveals two problems which must be solved before on can hope to get interesting results at finite N or in more interesting theories.

The first problem can be seen at lowest order, Eqs. (II.4), (II.5) and (II.6), which are true for all N: these equations do not give finite results for $\varepsilon \to 0$ at fixed finite N, due to the mismatch between the coefficient $(N-1)$ appearing in Eq. (II.6) and the coefficient $(N-1/2)$ appearing in Eq. (II.4): the limits $N \to \infty$ and $\varepsilon \to 0$ do not commute at this stage, and it will be the essential content of this paper and the next one to make these two limits commute, *i.e.* to find how to reconcile in a general case a variational ansatz with the fine tuning of renormalization, materialized by Eq. (II.6) and make the physics continuous as ε crosses zero at fixed renormalized coupling constant g^2 and mass scale μ.

The other problem appears at second order in Eq. (II.7), and is again related to the $\varepsilon \to 0$ limit. In this equation one finds that the quantity

$$E^{(0)}(m_0) \; - \; m_0 \frac{\partial E^{(0)}(m_0)}{\partial m_0} \; + \; \frac{1}{2} m_0^2 \frac{\partial^2 E^{(0)}(m_0)}{\partial m_0^2} \tag{II.8}$$

is finite as $\varepsilon \to 0$ although each term is separately divergent. Hence, in some sense, the lowest order in perturbation theory becomes irrelevant relative to the next two, each of order ε^{-1} (taking into account the fact that g_0^2 is of order ε). This fact is independent of the large-N limit, and generalizes to higher orders of the calculation: at any finite order, for any N, the last two orders dominate all previous ones as $\varepsilon \to 0$. This is quite unpleasant, as one would expect that all orders of perturbation theory contain at most a roughly equivalent amount of information. This is even more unpleasant in a perspective where one hopes to obtain accurate results from a low-order calculation!

These two pathologies of the $\varepsilon \to 0$ limit will be handled in the next section in the large-N limit.

Similar low-order calculations can be performed on the mass gap of the massive Lagrangian (II.1). The contributing cactus diagrams up to order g_0^4 are on Fig. 2 and, for all N, they give simply

$$m_F(m_0) = m_0 \; + \; (2N-1)\, g_0^2\, m_0^{1+\varepsilon} \frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}$$

$$+ (2N-1)^2\, g_0^4\, m_0^{1+2\varepsilon}\, \frac{\Gamma^2(-\frac{\varepsilon}{2})}{(4\pi)^{2+\varepsilon}}\, (1+\varepsilon) \;+\; O(g_0^6)$$

$$\equiv m_F^{(0)} \;+\; g_0^2\, m_F^{(1)} \;+\; g_0^4\, m_F^{(2)} \quad . \tag{II.9}$$

Perturbing at order 1 in both g_0^2 and m_0, one has the mass gap

$$m_F^{(1)}(m_0) = m_F^{(0)} \;+\; g_0^2\, m_F^{(1)} \;-\; m_0\, \frac{\partial m_F^{(0)}}{\partial m_0} \;=\; g_0^2\, m_F^{(1)} \quad . \tag{II.10}$$

This has no non trivial extremum in m_0, but gives the exact large-N mass gap when the variational value of Eq. (II.4) is used for m_0. Perturbing at order 2 in both g_0^2 and m_0 gives

$$m_F^{(2)}(m_0) = g_0^2\, m_F^{(1)} \;-\; g_0^2\, m_0\, \frac{\partial m_F^{(1)}}{\partial m_0} \;+\; g_0^4\, m_F^{(2)} \quad . \tag{II.11}$$

Again, this gives the exact large-N mass gap when the value (II.4) for m_0 is used, but has a very singular behaviour if one tries instead to extremize it with respect to m_0. Indeed, in the limit $\varepsilon \to 0$, the term $g_0^4\, m_F^{(2)}$ dominates, and it has no non trivial minimum.

These pathologies extend to higher finite orders of the large-N limit: plugging in the variational value (II.4) for m_0 always gives the exact large-N mass gap, but for $\varepsilon \to 0$, only the last order of perturbation theory survives, excluding any useful direct variational estimate of m_F; just as for the vacuum energy, physics does not seem smooth in the $\varepsilon \to 0$ limit, and we shall also solve this problem in the next section.

III. ALL ORDER CALCULATIONS AND RENORMALIZATION IN THE LARGE-N LIMIT.

In this section, we fully exploit the fact that the large-N limit can be solved in closed (implicit) form and use further properties of the variationally improved perturbation theory discovered in one dimension (Ref. [13]) to make renormalization transparent to our variational method, and the physics continuous as ε goes to zero.

For the vacuum energy density, the large-N limit amounts to retaining only the cactus vacuum diagrams. For the Lagrangian density of Eq. (II.1), this can be evaluated exactly and gives

$$E(m_0) = \frac{-m_0^2\, N}{2\, g_0^2\, [\, 1 - \frac{(2N-1)\, g_0^2\, m_F^\varepsilon\, \Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\,]\,(N-\frac{1}{2})} + \frac{m_0^2}{2\, g_0^2}\, \frac{N}{N-\frac{1}{2}}$$
$$+ \frac{\varepsilon}{2+\varepsilon}\, \frac{N\,\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\, m_F^{2+\varepsilon} \quad , \tag{III.12}$$

where m_F is given by

$$m_F = \frac{m_0}{1 - \frac{(2N-1)\, g_0^2\, m_F^\varepsilon\, \Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}} \quad . \tag{III.13}$$

Physically, in the usual treatment of the model, m_F is the mass gap, and is determined from this last equation, which has a non trivial solution in the symmetric limit $m_0 \to 0$. In the present variational method, we must take into account the perturbation in m_0 as shown in Eqs. (II.3) and (II.7). This is done rather easily in terms of contour integrals. Define a function $f(x)$ by

$$f(x) = 1 - \frac{(2N-1)\, g_0^2\, \Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\, x\, m_0^\varepsilon\, (1-x)^\varepsilon\, f^{-\varepsilon} \quad . \tag{III.14}$$

This function has power series expansions in both g_0^2 and x, beginning with 1.

The n^{th} order of perturbation theory for the vacuum energy density of the Lagrangian (II.1) is then:

$$E^{(n)}(m_0) = \frac{1}{2i\pi} \oint \frac{dx}{x}\, x^{-n}\, (1-x)\, [\, -\frac{m_0^2}{2\, g_0^2\, f} + \frac{m_0^2}{2\, g_0^2}\,]\, \frac{N}{N-\frac{1}{2}}$$
$$+ \frac{1}{2i\pi} \oint \frac{dx}{x}\, x^{-n}\, (1-x)^{1+\varepsilon}\, \frac{\varepsilon}{2+\varepsilon}\, \frac{N\,\Gamma(-\frac{\varepsilon}{2})\, m_0^{2+\varepsilon}}{(4\pi)^{1+\frac{\varepsilon}{2}}}\, f^{-2-\varepsilon} \quad , \tag{III.15}$$

where the integration is counterclockwise on a small circle around the origin.

It is of course straightforward to check that for $n = 2$ one precisely reproduces Eq. (II.7). It is also straightforward to check that

$$\frac{\partial E^{(n)}(m_0)}{\partial m_0} = 0 \tag{III.16}$$

at m_0 given by Eq. (II.4) and that the corresponding extremum of $E^{(n)}$ is independent of n, establishing in a few lines that the variational result is independent of the order of perturbation theory in the large-N limit. Indeed, for

$$1 = (2N-1)\, \frac{\Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}}\, g_0^2\, m_0^{\varepsilon} \quad , \tag{III.17}$$

one has simply

$$f(x) = 1 - x \quad . \tag{III.18}$$

Furthermore in general one may rewrite the definition of f, Eq. (III.14) as

$$f(x) = (1-x)\, [\, \frac{(4\pi)^{1+\frac{\varepsilon}{2}}}{(2N-1)\, g_0^2\, \Gamma(-\frac{\varepsilon}{2})}\,]^{-\frac{1}{\varepsilon}}\, (1-f)^{-\frac{1}{\varepsilon}} \quad , \tag{III.19}$$

which shows that at fixed $\varepsilon < 0$, f can be expanded in a power series of $(1-x)$. After a numerical exploration of the complex plane to make sure that no extra singularities lie in the way, one may distort the integration contour in Eq. (III.15) to run clockwise around the cut lying along the real positive axis and starting at $x = 1$. Actually, in Eq. (III.15), $x = 1$ is an isolated pole, whose residue dominates its large n behaviour; this immediately shows that in the large-N limit, at fixed ε for any fixed m_0, $E^{(n)}(m_0)$ goes to the exact answer Eq. (II.5), in the limit of infinite order. This is reassuring: in the large N case, which is well under control, and where ordinary perturbation theory has a finite radius of convergence, the split of Ref. [12,13] between free and interacting terms is indeed independent of m_0 in the limit of infinite order, as intuition would suggest.

However one could go one step further, and, as already noticed in the case of the anharmonic oscillator in Ref. [13], extract more structure from the limit of infinite order by rescaling m_0 with the order n.

After distortion of the contour it is clear that only the vicinity of $x = 1$ survives in the limit $n \to \infty$, and one analyses this by the change of variable

$$1 - x = \frac{v}{n} \quad . \tag{III.20}$$

Rescaling m_0 by introducing $m_0 = m'_0\, n$, m'_0 kept fixed as n goes to infinity, one finds that $E^{(n)}$ in Eq. (III.15) has a limit $E(m'_0)$ given by

$$\begin{aligned} E(m'_0) = & \frac{1}{2i\pi} \oint v\, dv\, e^v \, [- \frac{{m'_0}^2}{2\, g_0^2\, f_1(v)} \; + \; \frac{{m'_0}^2}{2\, g_0^2} \,] \, \frac{N}{N - \frac{1}{2}} \\ & + \frac{1}{2i\pi} \oint v\, dv\, e^v \, \frac{\varepsilon\, N\, \Gamma(-\frac{\varepsilon}{2})\, {m'_0}^{2+\varepsilon}}{(2+\varepsilon)\, (4\pi)^{1+\frac{\varepsilon}{2}}} \, f_1(v)^{-2-\varepsilon} \quad , \end{aligned} \tag{III.21}$$

with f_1 defined simply by

$$f_1(v) = 1 - \frac{(2N-1)\, g_0^2\, \Gamma(-\frac{\varepsilon}{2})}{(4\pi)^{1+\frac{\varepsilon}{2}}} \, (m'_0\, v)^\varepsilon \; f_1(v)^{-\varepsilon} \quad , \tag{III.22}$$

and the v integration contour lying counterclockwise around the negative real axis in the cut v-plane.

The function $E(m'_0)$ has the remarkable property, not shared by $E^{(n)}(m_0)$, to allow for a smooth limit as $\varepsilon \to 0$. This follows from the fact that Eq. (III.22) which defines f_1 is continuous as $\varepsilon \to 0$ provided that g_0^2 and m_0 follow their renormalization group behaviour. In this limit, we obtain simply

$$E(m'_0) = E(m', g^2, \mu) = \frac{{m'}^2}{2i\pi} \oint \frac{v\, dv\, e^v}{f_2^2} \, \frac{N}{4\pi} \, (\, 1 \; + \; \frac{2\, f_2\, \pi}{(N - \frac{1}{2})\, g^2} \,) \quad , \tag{III.23}$$

with

$$\begin{aligned} f_2\,(v) = 1 \; + \; \frac{g^2\, N}{\pi} \, \ln \frac{m'\, v}{\mu\, f_2} \; - \; \frac{N\, g^2}{2\pi} (\, \gamma_E \; + \; \ln 4\pi \,) \\ \gamma_E = 0.577215... \quad , \end{aligned} \tag{III.24}$$

where we have introduced the renormalized mass

$$m'_0 = \frac{m'}{1 - \frac{N\,g^2}{\pi\,\varepsilon}} \quad . \tag{III.25}$$

Eq. (III.23) involves only renormalized quantities, and is renormalization group invariant in the sense of the effective massive theory:

$$[\,\mu\,\partial_\mu \; + \beta(g)\,g\,\frac{\partial}{\partial\,g} \; - \; \gamma(g)\,m\,\frac{\partial}{\partial\,m}\,]\;E(m', g^2, \mu) = 0 \quad , \tag{III.26}$$

with

$$\beta(g) = -\,\frac{g^2\,N}{2\pi} \quad , \quad \gamma(g) = \frac{g^2\,N}{\pi} \quad . \tag{III.27}$$

This follows trivially from the fact that $E(m', g^2, \mu)$ depends only on the bare parameters m'_0 and g_0^2. What is much less trivial is that we have arrived at a finite quantity for $\varepsilon \to 0$: $E^{(n)}(m_0)$ in Eq. (III.15) is also renormalization group invariant but not finite in the limit $\varepsilon \to 0$, while $E(m'_0)$ is both invariant and finite.

One can go one step further by defining the dimensionless parameter

$$m'' = \frac{m'\,\pi}{N\,g^2\,\mu\,e^{-\frac{\pi}{N g^2}}} \tag{III.28}$$

and a function f_3 by

$$f_3 = \ln(\,m''\,v\,) \; - \; \ln f_3 \quad . \tag{III.29}$$

m'' is a finite pure number invariant by the renormalization group of the effective theory.

Studying the function $E(m'')$ poses no problem. One finds that the function f_3 defined by Eq. (III.29) behaves as $\ln m'' + O(\ln\ln m'')$ as m'' goes to infinity, a typical renormalization groupimproved high energy behaviour. Furthermore, f_3 can be expanded in power series of v around $v = 0$, with radius of convergence $1/e$, and with a cut extending from $-1/e$ (a square root branch point) to $-\infty$. The behaviour of $E(m'')$ for large m'' can be obtained by standard methods of contour integrals, and is of the form

$$E(m'') \underset{m''\to\infty}{\sim} \frac{m''^2}{\ln m''} \quad . \tag{III.30}$$

This is the perturbative regime. For $m'' \to 0$, $E(m'')$ converges exponentially to

$$E(0) = -\frac{N}{4\pi}\,\mu^2\, e^{-\frac{2\pi}{Ng^2}} \quad , \tag{III.31}$$

which is the only real extremum. Plotting the curve $E(m'')$ poses no problem with a workstation, and the result can be seen on Fig. 3. Notice its remarkably smooth behaviour.

The extremum at $m'' = 0$ is the exact value derived from the original calculation using the effective potential of the field $\sigma = \bar{\psi}\psi$. This fact is consistent with the rescaling $m_0 = m'_0\, n$ as the order goes to infinity, and the fixed position of the extremum of $E^{(n)}$ given by Eq. (II.4).

The same procedure can be applied to the calculation of the mass gap. From Eq. (III.13), the n^{th} order of perturbation theory for the mass gap is

$$m_F^{(n)} = \frac{1}{2i\pi} \oint \frac{dx}{x}\, x^{-n}\, \frac{m_0}{f} \quad . \tag{III.32}$$

This reproduces the low order calculation (II.9); when m_0 is the value of Eq. (II.4) which extremises $E^{(n)}$, it is immediate that $m_F^{(n)} = m_0$; one may also go to infinite order, following the same rescaling of $1-x$ and m_0 as for the vacuum energy density. In this infinite order limit,one obtains

$$m_F(m', g^2, \mu) = \frac{m'}{2i\pi} \oint \frac{dv\, e^v}{f_2} \quad , \tag{III.33}$$

a formula where the $\varepsilon \to 0$ limit has been taken like in the case of $E(m', g^2, \mu)$, and contains the variational parameter m' in a finite renormalization group invariant way.

$m_F(m', g^2, \mu)$ is actually a function of the renormalization invariant parameter m'' only; $m_F(m'')$ can be analyzed like $E(m'')$, approaching exponentially the exact value at $m'' = 0$, its only real extremum. It is plotted in Fig. 4.

IV. CONCLUSION.

In this paper, with the Gross-Neveu model in the large N limit as example, we have studied some pathologies of the variational method applied to a renormalizable quantum field theory. On the two examples of the vacuum energy density and the mass gap, we have shown how a resummation to all orders can make these pathologies disappear, and make a variational method perfectly compatible with renormalization, *i.e.* one may replace the original theory (the massless model) by another one (a massive model), varying the parameters while keeping physics continuous as the space-time dimension is varied around its critical value. In the next paper, we extend the procedure to the finite-N case, where the mathematics of the renormalization group and the Feynman diagram stuctures are more generic.

V. ACKNOWLEGMENTS

One of us (A. N.) is grateful for their hospitality to the Lawrence Berkeley Laboratory, where this work was supported in part by the Director, Office of Energy research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and to the Physics Department, University of California, Berkeley, where this work was completed with partial support from National Science Foundation grant PHY-90-21139. (C.A.) is grateful to the theory group of Imperial College for their hospitality and acknowledges financial support from the Ministère de la Recherche et de la Technologie, and from E.E.C. grant No. ERBCHBICT941235.

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Figure Captions.

Figure 1. Vacuum energy graphs in the large-N limit.

Figure 2. Mass gap graphs in the large-N limit.

Figure 3. Variational vacuum energy $E(m'')/E(0)$ as a function of m''.

Figure 4. Mass gap $m_F(m'')/m_F(0)$ as a function of the variational parameter m''.

FIGURES

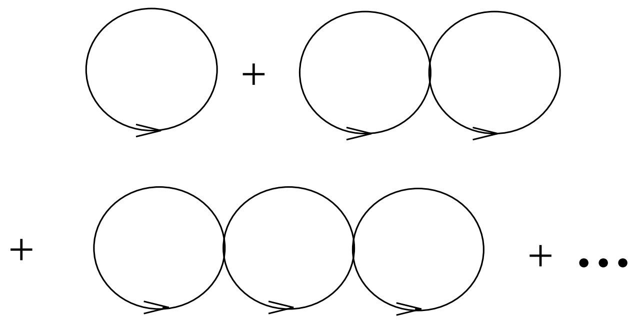

FIG. 1. Vacuum energy graphs in the large-N limit.

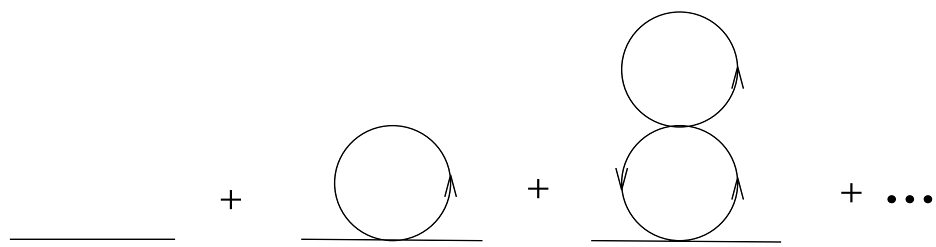

FIG. 2. Mass gap graphs in the large-N limit.

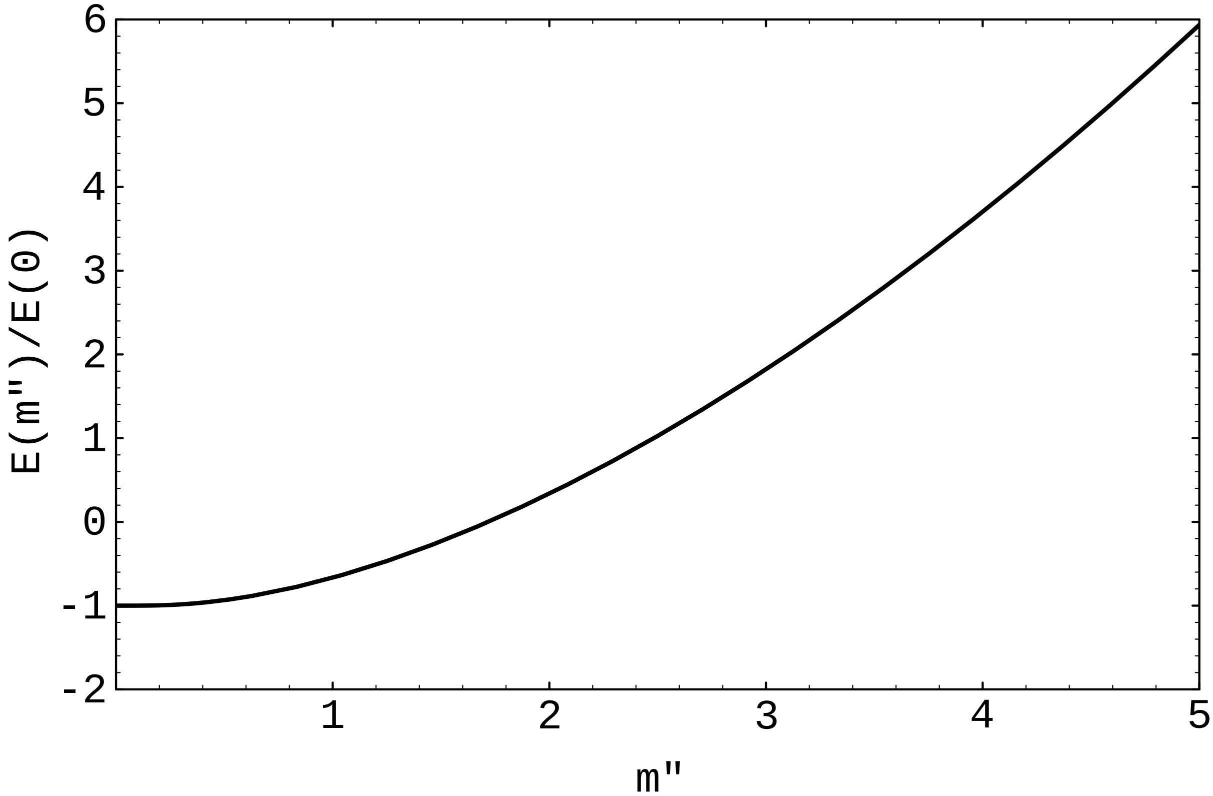

FIG. 3. Variational vacuum energy $E(m'')/E(0)$ as a function of m''.

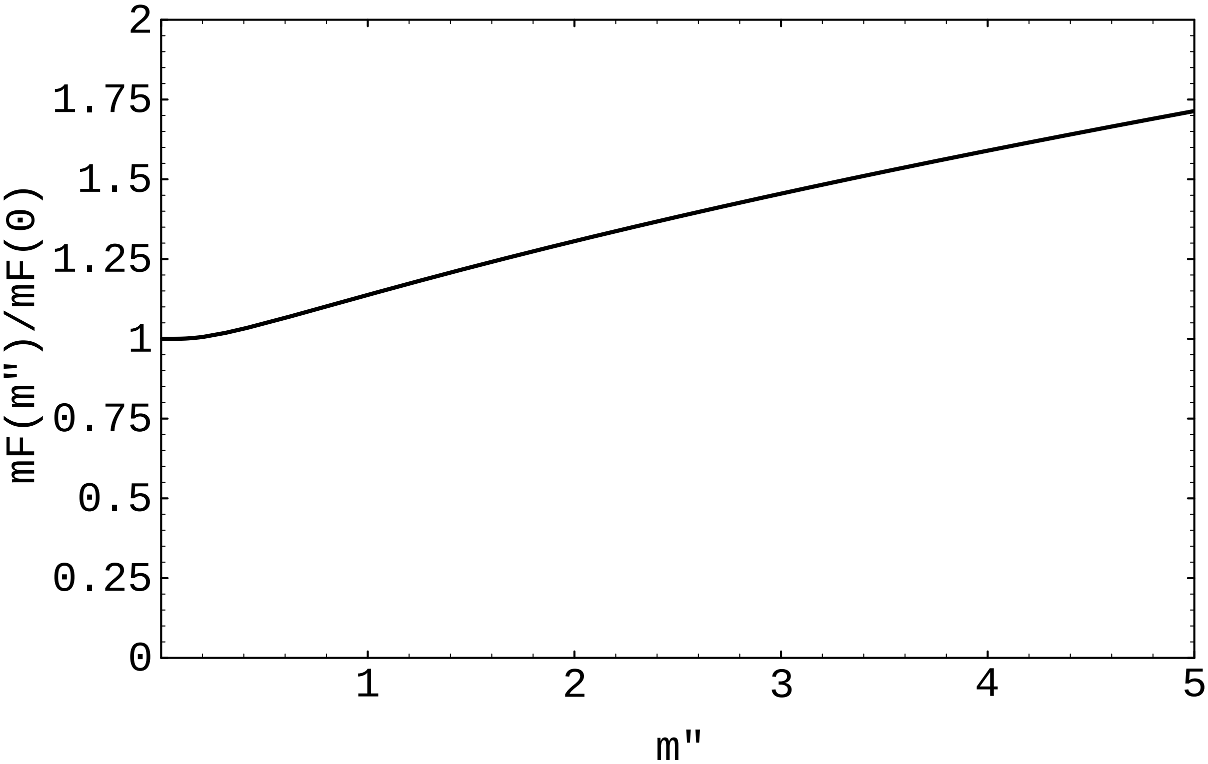

FIG. 4. Mass gap $m_F(m'')/m_F(0)$ as a function of the variational parameter m''.